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ON THE BUCKLING OF THICK SPHERICAL SHELLS UNDER NORMAL PRESSURE

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Abstract—Solutions of Bolotin's equations are found for the buckling of thick spherical shells under pressure. Legendre polynomials are used to express the meridional variation of displacement. The radial variation of displacement is expressed by polynomials in terms of inverse powers of the radius, the coefficients of the series being determined from simultaneous equations. A modified form of the boundary conditions, which allows for change of surface area as well as the surface rotation considered by Bolotin, is used. Results are compared with the classical theory for thin shells.

1. INTRODUCTION

The classic solution for the critical buckling load of a thin-walled spherical shell under external pressure was derived by Zoelly[1]. The resulting formula for the critical pressure p_c can be written as

$$\frac{p_c}{G} = 16\left[\frac{b-a}{b+a}\right]^2\left[\frac{1+\nu}{3(1-\nu)}\right]^{1/2} \tag{1}$$

where G is the shear modulus, a and b the internal and external radii, and ν is Poisson's ratio. For the purposes of comparison with the following analysis, p_c will be taken more generally as the difference between the external and internal pressure.

In considering the buckling of a shell of finite thickness, it is possible to use a number of formulations for the infinitesimal incremental deformation of initially stressed continuous bodies. Such formulations have been summarized by Bazant[2]. One of the problems which arise is that if a strain-energy density function is postulated for the material in terms of Lagrangian strain which initially corresponds to linear, isotropic behaviour, the relationships between increments of stress and strain when some initial critical stress is reached cannot correspond exactly to a linear isotropic response. However, it must be born in mind that the engineering theory of elastic stability is based on just such an assumption, as are the equations of Bolotin[3] which will be used here. It has been shown by the author[4, 5] that these lead to solutions for struts and cylinders which are closely related to standard engineering solutions. Babich[6], starting from slightly different fundamental equations, also obtained solutions for the cylinder problem, as did Demiray[7], Green and Spencer[8] and Wilkes[9] for incompressible materials.

Various solutions for the buckling of a spherical shell under pressure have been presented where different strain energy density functions of a neo-Hookean type have been postulated and incompressibility assumed. Wesolowski[10] examines the behaviour of a Mooney material and solves the equations derived by means of finite difference approximations to the differentials. Wang and Ertepinar[11] also solve problems related to the stability and vibration of thick cylindrical and spherical shells using finite difference methods. As pointed out by Hill[12] such methods could give rise to large numerical errors and he proposed a strain energy density function which leads to a closed form of solution which requires the evaluation of integrals of functions related to the radial variation of displacement. By solving Bolotin's equations ([13], eqn 13.16) and fitting the solutions to rather more carefully considered boundary conditions, the only material properties which need to be specified are the shear modulus and Poisson's ratio. As will be seen, no assumptions of incompressible behaviour are necessary.

2. THE CONTINUUM EQUATIONS

It has been shown[4] that it is sometimes convenient to write Bolotin's equations in the form

$$[(1-2\nu)\nabla\cdot\nabla+\alpha]\mathbf{u}+\nabla(\nabla\cdot\mathbf{u})=0 \tag{2}$$

where

$$\alpha = \frac{1-2\nu}{G}\left[\frac{\partial}{\partial x_j}\left(\sigma^*_{jk}\frac{\partial}{\partial x_k}\right) - \rho\frac{\partial^2}{\partial t^2}\right] \tag{3}$$

u is the displacement vector from the loaded state with initial stress σ^*_{jk} related to the cartesian material co-ordinates x_i, ρ is the mass per unit unstressed volume and the summation convention applies in eqn (3).

The expressions obtained for the radial stress σ_{rr} and tengential stresses $\sigma_{\theta\theta}$ and $\sigma_{\phi\phi}$ induced in a spherical shell by an internal pressure p_i and an external pressure p_0 are given by

$$\frac{\sigma_{rr}}{G} = 2p\frac{a^3}{r^3} + q \tag{4}$$

$$\frac{\sigma_{\theta\theta}}{G} = \frac{\sigma_{\phi\phi}}{G} = -p\frac{a^3}{r^3} + q \tag{5}$$

where

$$p = \frac{p_0 - p_i}{2G(1 - a^3/b^3)} \tag{6}$$

$$q = \frac{p_i a^3/b^3 - p_0}{G(1 - a^3/b^3)}$$

and the spherical co-ordinate system r, θ, ϕ is used. The displacement **u** can be expressed in terms of these co-ordinates by

$$\mathbf{u} = u\hat{\mathbf{r}} + v\hat{\boldsymbol{\theta}} + w\hat{\boldsymbol{\phi}} \tag{7}$$

where $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ are vectors in the directions of increasing r, θ and ϕ respectively. After multiplying by a factor of r^2, eqn (2) can now be written with respect to these co-ordinates as

$$\begin{aligned}(1-2\nu)&\left\{\left(q+1-p\frac{a^3}{r^3}\right)\left[r^2\Delta u - 2u - \frac{2}{\sin\theta}\left(\frac{\partial}{\partial\theta}(v\sin\theta) + \frac{\partial w}{\partial\phi}\right)\right] + 3p\frac{a^3}{r}\frac{\partial^2 u}{\partial r^2}\right.\\ &\left.-\frac{\rho r^2}{G}\frac{\partial^2 u}{\partial t^2}\right\} + r^2\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u)\right)\\ &+\frac{r^2}{\sin\theta}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial\theta}(v\sin\theta) + \frac{1}{r}\frac{\partial w}{\partial\phi}\right] = 0\end{aligned} \tag{9}$$

$$\begin{aligned}(1-2\nu)&\left\{\left(q+1-p\frac{a^3}{r^3}\right)\left[r^2\Delta v + 2\frac{\partial u}{\partial\theta} - \frac{1}{\sin^2\theta}\left(v + 2\cos\theta\frac{\partial w}{\partial\phi}\right)\right]\right.\\ &\left.+3p\frac{a^3}{r}\frac{\partial^2 v}{\partial r^2} - \frac{\rho r^2}{G}\frac{\partial^2 v}{\partial t^2}\right\} + \frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial\theta}\right)\\ &+\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}(v\sin\theta) + \frac{\partial w}{\partial\phi}\right] = 0\end{aligned} \tag{10}$$

$$\begin{aligned}(1-2\nu)&\left\{\left(q+1-p\frac{a^3}{r^3}\right)\left[r^2\Delta w + \frac{1}{\sin^2\theta}\left(2\sin\theta\frac{\partial u}{\partial\phi} + 2\cos\theta\frac{\partial v}{\partial\phi} - w\right)\right]\right.\\ &\left.+3p\frac{a^3}{r}\frac{\mathrm{d}^2 w}{\mathrm{d}r^2} - \frac{\rho r^2}{G}\frac{\mathrm{d}^2 w}{\mathrm{d}t^2}\right\} + \frac{1}{r\sin\theta}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial\phi}\right)\\ &+\frac{1}{\sin^2\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\phi}\right) + \frac{\partial^2 w}{\partial\phi^2}\right] = 0\end{aligned} \tag{11}$$

where

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2 \sin^2\theta}\left\{\sin\theta \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) + \frac{\partial^2}{\partial\phi^2}\right\}. \tag{12}$$

In this paper, rotationally symmetric solutions to the static buckling problem will be sought of the form

$$u = U(s)P_m(\cos\theta) \tag{13}$$

$$v = V(s)\frac{\partial P_m}{\partial\theta} \tag{14}$$

$$w = 0 \tag{15}$$

where

$$s = a/r \tag{16}$$

and $P_m(\cos\theta)$ is the mth Legendre polynomial. On substituting these expressions into eqn (9) and removing a common factor of P_m, it becomes

$$(1-2\nu)\left\{(q+1-ps^3)\left[\left(s^2\frac{d^2}{ds^2} - m^2 - m - 2\right)U + 2(m^2+m)V\right] + 3ps^3\left(s^2\frac{d^2}{ds^2} + 2s\frac{d}{ds}\right)U\right\} + \left(s^2\frac{d^2}{ds^2} - 2\right)U + (m^2+m)\left(s\frac{d}{ds}+1\right)V = 0 \tag{17}$$

and on substituting these expressions into eqn (10) and removing a common factor of $\partial P_m/\partial\theta$ it becomes

$$(1-2\nu)\left\{(q+1-ps^3)\left[\left(s^2\frac{d^2}{ds^2} - m^2 - m\right)V + 2U\right] + 3ps^3\left(s^2\frac{d^2}{ds^2} + 2s\frac{d}{ds}\right)V\right\} + \left(2 - s\frac{d}{ds}\right)U - (m^2+m)V = 0. \tag{18}$$

Equation (11) is automatically satisfied, the l.h.s. becoming identically zero.

Seeking solutions of the form

$$U = s^\beta \sum_{i=0}^{\infty} u_i s^{3i} \tag{19}$$

$$V = s^\beta \sum_{i=0}^{\infty} v_i s^{3i} \tag{20}$$

leads to indicial equations which are satisfied by each of the columns in Table 1.

Table 1. Solutions of the indicial equations

Series	U_1, V_1	U_2, V_2	U_3, V_3	U_4, V_4
β	$-m-1$	$-m+1$	m	$m+2$
u_0	$(m+1)[m-2(1-2\nu)(q+1)]$	m	$m[2(1-2\nu)(q+1)+m+1]$	$-m-1$
v_0	$2(1-2\nu)(q+1)+m+3$	1	$2(1-2\nu)(q+1)+2-m$	1

The recurrence relationship is given by the simultaneous equations

$$\begin{aligned}&\{[1-2\nu)(q+1)+1](\beta+3i+1)(\beta+3i-2)-(1-2\nu)(q+1)(m^2+m)\}u_i\\&\quad+(m^2+m)\times[2(1-2\nu)(q+1)+\beta+3i+1]v_i\\&\quad+(1-2\nu)p\{[2(\beta+3i-1)(\beta+3i-3)+m^2+m+2]u_{i-1}-2(m^2+m)v_{i-1}\}=0\end{aligned} \tag{21}$$

$$\begin{aligned}&[2(1-2\nu)(q+1)+2-(\beta+3i)]u_i+\{(1-2\nu)(q+1)[(\beta+3i)(\beta+3i-1)-m^2-m]\\&\quad-m^2-m\}v_i+(1-2\nu)p\{-2u_{i-1}+[2(\beta+3i-3)(\beta+3i-1)+m^2+m]v_{i-1}\}=0.\end{aligned} \tag{22}$$

Problems arise when there are terms in the third and fourth solutions which have the same powers of s as those in the first and second solutions. Three cases can be defined in terms of the values of m in relation to some general integer n.

(I) $m=3n$; the fourth solution interferes with the first.

(IIa) $m=3n+1$; the third solution interferes with the first.

(IIb) $m=3n+1$; the fourth solution interferes with the second.

(III) $m=3n+2$; the third solution interferes with the second.

In all cases critical conditions, when interference first occurs in a series, are when i is $2n+1$. It is necessary in these cases to introduce the equivalent form of solution to that used for single differential equations under such circumstances. This can be written as

$$U=s^\beta\sum_{i=0}^{\infty}u_is^{3i}+c\log s\quad U^* \tag{23}$$

$$V=s^\beta\sum_{i=0}^{\infty}v_is^{3i}+c\log s\quad V^* \tag{24}$$

where U^* and V^* are the interfering solutions. This gives additional terms on the l.h.s. of eqn (21) of the form

$$\begin{aligned}&c\{(2-2\nu)(q+1)[2(\beta+3i)-1]u^*_{i-2n-1}+(m^2+m)v^*_{i-2n-1}\\&\quad+4(1-2\nu)p(\beta+3i-2)u^*_{i-2n-2}\}\end{aligned} \tag{25}$$

and additional terms on the l.h.s. of eqn (22) of the form

$$c\{(1-2\nu)(q+1)[2(\beta+3i)-1]v^*_{i-2n-1}-u^*_{i-2n-1}+4(1-2\nu)p(\beta+3i-2)v^*_{i-2n-2}\}. \tag{26}$$

The value of c is found from the critical equations (when i equals $2n+1$). Under these circumstances, the linear expression in u_i and v_i in eqns (22) is a multiple of the expression in eqn (23) so that both u_i and v_i can be eliminated simultaneously by adding suitable multiples of these equations together. The resulting equation determines the value of c. The necessary multipliers are gives in Table 2.

Having found c, an arbitrary value, such as zero, can be assigned to u_{2n+1} or v_{2n+1} and the other coefficient found from the augmented form of eqn (21) or eqn (22). Any other arbitrary choice will differ by some multiple of the interfering solution. Since the complete set of complementary functions is sought, the general nature of result is not affected.

Table 2. Equation multipliers to determine c

Case	Equation (21) multiplier	Equation (22) multiplier
I	$m-2(1-2\nu)(q+1)$	$m[2(1-2\nu)(q+1)+m+3]$
IIa	1	m
IIb	$m-1-2(1-2\nu)(q+1)$	$(m-1)[2(1-2\nu)(q+1)+m+2]$
III	1	$m+1$

3. THE BOUNDARY CONDITIONS

The boundary conditions for this problem depend on the increment $\Delta \mathbf{p}$ of the surface traction which arises in moving from the initially loaded state to the deflected (buckled) state. This is related by Bolotin ([3], eqn 13.17) to the increments in displacement. The radial and tangential components of this traction for the problem under discussion are given by

$$\frac{\Delta p_r}{G} = (2ps^3 + q)\frac{\partial u}{\partial r} + 2\frac{\partial u}{\partial r} + \frac{2\nu}{r(1-2\nu)}\left\{2u + r\frac{\partial u}{\partial r} + \frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}(v\sin\theta) + \frac{\partial w}{\partial\phi}\right]\right\} \tag{27}$$

$$\frac{\Delta p_\theta}{G} = (2ps^3 + q)\frac{\partial v}{\partial r} + \frac{1}{r}\left(\frac{\partial u}{\partial\theta} - v + r\frac{\partial v}{\partial r}\right) \tag{28}$$

$$\frac{\Delta p_\phi}{G} = (2ps^3 + q)\frac{\partial w}{\partial r} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial u}{\partial\phi} - w + r\frac{\partial w}{\partial r}\right). \tag{29}$$

For the deflection mode chosen, the component in the direction of $\hat{\boldsymbol{\phi}}$ is identically zero. The deflection of an element of surface is shown in Fig. 1. The rotation γ is given by

$$\gamma = v/(r+u) \doteqdot v/r. \tag{30}$$

The changes between the initial and final states are found from
the original unit normal:

$$\hat{\mathbf{r}}$$

the final unit normal:

$$\hat{\mathbf{r}} + \left(\sin\gamma - \frac{1}{r}\frac{\partial u}{\partial\theta}\right)\hat{\boldsymbol{\theta}}$$

$$\doteqdot \hat{\mathbf{r}} + \frac{1}{r}\left(v - \frac{\partial u}{\partial\theta}\right)\hat{\boldsymbol{\theta}}$$

the original elementary area:

$$r\sin\theta\,\delta\phi\,r\,\delta\theta$$

the final elementary area:

$$(r+u)\sin(\theta+\gamma)\delta\phi(r+u)(\delta\theta+\delta\gamma)$$

$$\doteqdot r\sin\theta\,\delta\theta\,\delta\phi\left[r + 2u + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v\sin\theta)\right].$$

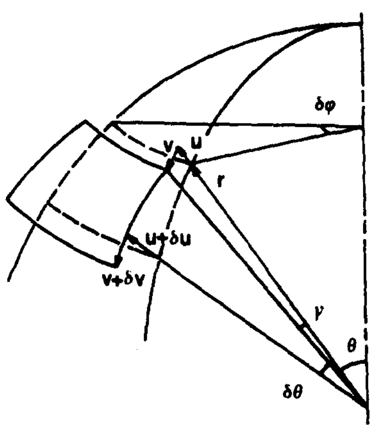

Fig. 1. Displacement of a surface element.

Then if the normal pressure is p_0,

$$\Delta\mathbf{p}(r^2 \sin\theta\delta\theta\delta\phi) = -p_0 r \sin\theta\delta\theta\delta\phi \left[r + 2u + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v\sin\theta)\right] \times \left[\hat{\mathbf{r}} + \frac{1}{r}\left(v - \frac{\partial u}{\partial\theta}\right)\hat{\boldsymbol{\theta}}\right] - r\hat{\mathbf{r}}\bigg\} \tag{31}$$

which gives, to the required order of accuracy,

$$\Delta p_r = \frac{-p_0}{r}\left[2u + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v\sin\theta)\right] \tag{32}$$

$$\Delta p_\theta = \frac{-p_0}{r}\left[v - \frac{\partial u}{\partial\theta}\right]. \tag{33}$$

The appropriate expressions for an internal surface are found by substituting p_i for p_0. It should be noted that these are not the same expressions as those which would be obtained for Bolotin's hydrostatic loading ([3], eqn 13.18 *et seq.*). This is because he makes no allowance for the change in area of an elementary surface element, which produces effects of the same order of magnitude as its rotation. Using eqns (6), (7), (13), (14), (27), (28), (32) and (33) gives the exterior boundary conditions as

$$2\left(1 - \frac{p_0}{G}\right)s\frac{\mathrm{d}U}{\mathrm{d}s} + \left(\frac{2\nu}{1-2\nu} + \frac{p_0}{G}\right)\left[s\frac{\mathrm{d}U}{\mathrm{d}s} - 2U + (m^2+m)V\right] = 0 \tag{34}$$

$$\left(1 - \frac{p_0}{G}\right)\left[U - V - s\frac{\mathrm{d}V}{\mathrm{d}s}\right] = 0 \tag{35}$$

where a common factor of P_m has been removed from eqn (34) and a common factor of $\partial P_m/\partial\theta$ removed from eqn (35). The internal surface boundary conditions are given by replacing p_0 by p_i in these two equations. Taking U and V to be a linear combination of the four functions found,

$$U = \sum_{j=1}^{4} A_j U_j \tag{36}$$

$$V = \sum_{j=1}^{4} A_j V_j \tag{37}$$

and substituting these expression into the above boundary conditions gives four simultaneous equations for the constants A_j. For these to be non-zero, the determinant of the coefficients of A_j in these equations must be zero. This condition gives the instability criterion from which the buckling pressures are determined. It should be noted that eqn (35) and its equivalent for the internal surface imply possible buckling pressures of

$$p_o = G \tag{38}$$

and

$$p_i = G. \tag{39}$$

This condition occurs in a number of other problems, for example it corresponds to the torsional buckling pressure for an axially-loaded circular cylinder[5], which is also when the reduced torsional stiffness of the section, according to Wagner's hypothesis, becomes zero. However, it has been suggested that this might be an artefact of the engineering theory of buckling.

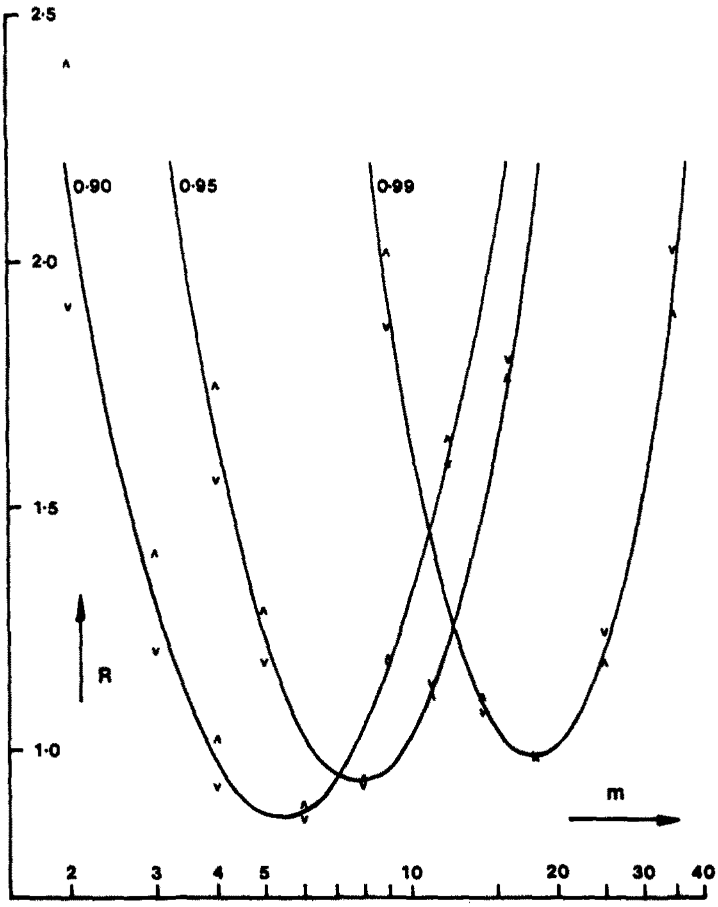

Fig. 2. Variation of critical load ratio with frequency.

4. DISCUSSION

The critical buckling pressures were found from the zeros of the determinant derived in the previous section. This was done using a desk computer in an interactive mode so that series convergence could be checked. For relatively thin shells at low values of m this is extremely rapid but some series become divergent in the region of the critical pressure when a/b is less than about 0.55 or p_i/p_0 is greater than unity. The graphs shown plot the ratio of the critical pressure difference found to that given by eqn (1). In Figs. 2 and 3 the internal pressure is assumed to be zero.

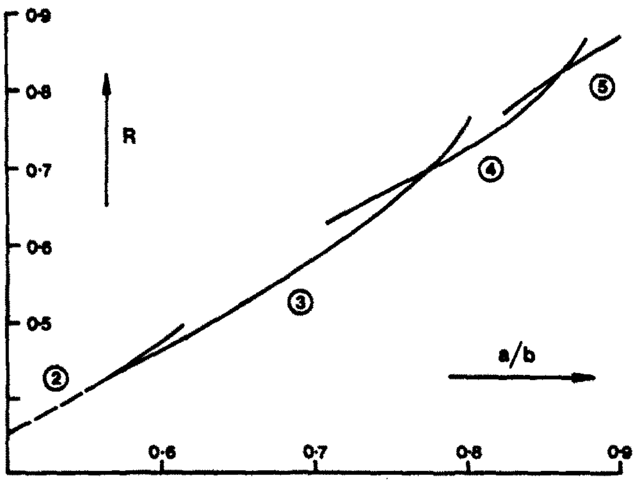

Fig. 3. Variation of critical load ratio with thickness.

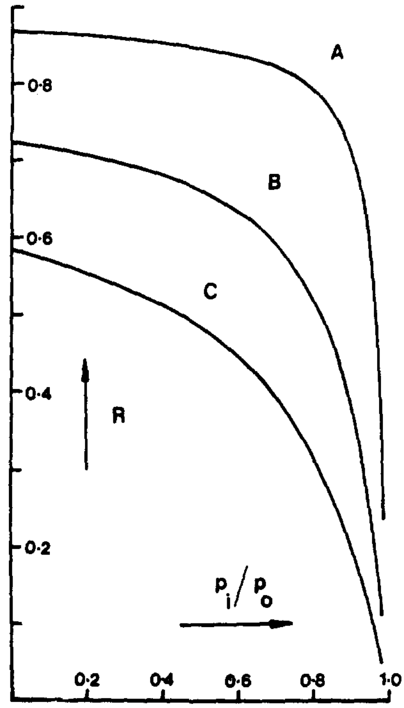

Fig. 4. Variation of critical load ratio with pressure ratio.

Figure 2 shows the variation of critical pressure ratio with m for shell thickness ratios, a/b, of 0.90, 0.95 and 0.99, these values being marked by the appropriate curve. The mid-range value of 0.25 was chosen for Poisson's ratio and results for ratios of 0.4 and 0.1 are marked by the apices of vees and inverted vees respectively. It will be seen that the ratio R does not always vary in the same way with Poisson's ratio, nor does it even vary monotonically. For example, when a/b is 0.95 and m is 16, the result for Poisson's ratio equal to 0.25 lies outside the range between the two extreme values. Further analysis shows that the minimum value of R (1.767) occurs for a Poisson's ratio of 0.178 in this instance. The value of m for the lowest critical pressure increases with decreasing shell thickness, as predicted by thin shell theory.

Results for thicker shells are shown in Fig. 3. Again a Poisson's ratio of 0.25 has been chosen and only the curves for the values of m producing the lowest critical pressure are drawn. These values are marked by the curves. Below a/b equal to 0.9, the m values are greater than or equal to those found by Hill[12] and Ertepinar[13] and above this value they are greater than or equal to those of Hill but less than those of Ertepinar. It must be borne in mind that these two authors obtained solutions for incompressible materials and that slight changes in the curves on Fig. 3 will change the intersection points considerably.

Figure 4 shows the sensitivity of the shells to the pressure ratio p_i/p_0, the classical result given by eqn (1) depending only on the pressure difference. It will be seen from eqns (38) and (39) that a solution exists even when the pressure difference is zero, so that these curves can pass through the point (1, 0). All the curves are drawn for Poisson's ratio equal to 0.25. Curve A is for a/b equal to 0.9 and m equal to 5. Curves B and C are for a/b equal to 0.8 and 0.7 and m equal to 4 and 3 respectively. Clearly, the thicker shell is, the more sensitive it becomes to the pressure ratio. The possibility of an inflation instability, when the internal pressure exceeds the external pressure, has been discussed by Durban and Baruch[14] for incrementally elastic thick-walled spheres. Equation (39) gives an upper bound to their results. Similar methods can be used to find buckling pressures for thick cylinders[15].

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